# MATH 5061 Lecture 10 (Mar 24)

[Problem Set 5 posted, due on Apr. 7.]

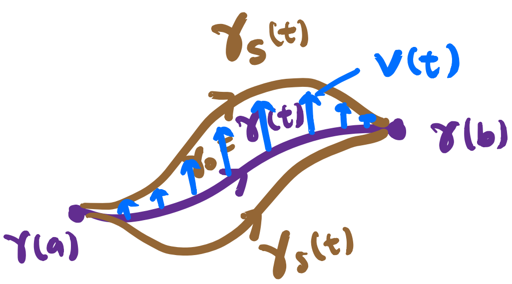
Last time...... $1^{st}$ & $2^{nd}$ variation of length / energy

Given a curve $\gamma^{(t)}: [a,b] \to (M^n, g)$, defined

$$E(\gamma) := \frac{1}{2}\int_a^b g(\gamma', \gamma')\, dt$$ energy ← dep. on parametrization

$$L(\gamma) := \int_a^b \sqrt{g(\gamma', \gamma')}\, dt$$ length ← indep. of parametrization

Consider a 1-parameter variation of curves (with fixed end points).




$\gamma(t,s) = \gamma_s(t) : [a,b] \times (-\varepsilon, \varepsilon) \longrightarrow M$ smooth

$\leadsto$ $E(\gamma_s)$, $L(\gamma_s)$ are smooth fcn of $s$

compute $\left.\frac{d}{ds}\right|_{s=0} E(\gamma_s)$ and $\left.\frac{d^2}{ds^2}\right|_{s=0} E(\gamma_s)$

variation field

$$V(t) := \left.\frac{\partial \gamma}{\partial s}\right|_{s=0}$$

Prop: ($1^{st}$ variation formula for energy)

$$E'(s) = -\int_a^b \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle dt$$

At a critical pt $\gamma_0$ for $E$, i.e. a geodesic, then we compute

Prop: ($2^{nd}$ variation formula for energy)

$$E''(0) = \int_a^b \langle \nabla_{\frac{\partial}{\partial t}} V, \nabla_{\frac{\partial}{\partial t}} V \rangle - \langle R(\tfrac{\partial \gamma}{\partial t}, V)\tfrac{\partial \gamma}{\partial t}, V \rangle\, dt$$

Proof: Recall:

$$E'(s) = -\int_a^b \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle dt$$

Differentiate w.r.t. $s$, evaluate at $s=0$.

$$E''(0) = \frac{d}{ds}\Big|_{s=0}\left(-\int_a^b \left\langle \frac{\partial\gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial t}\right\rangle dt\right)$$

$$= \frac{d}{ds}\Big|_{s=0}\left(\int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right\rangle dt\right)$$

metric compatible

$$= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial s}, \boxed{\nabla_{\frac{\partial}{\partial s}}\frac{\partial\gamma}{\partial t}}\right\rangle + \left\langle \nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right\rangle dt$$

|| torsion-free

swap

$$= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial s}, \boxed{\nabla_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial s}}\right\rangle dt$$

$$+ \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right\rangle + \left\langle R\left(\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s}\right)\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right\rangle dt$$

at $s=0$

$$= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}}V, \nabla_{\frac{\partial}{\partial t}}V\right\rangle - \left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial\gamma}{\partial s}, \cancel{\nabla_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial t}}\right\rangle - \left\langle R\left(\frac{\partial\gamma}{\partial t}, V\right)\frac{\partial\gamma}{\partial t}, V\right\rangle dt$$

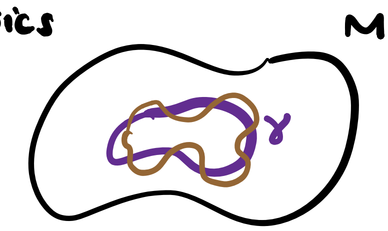
$=0$ $\because \gamma_0$ is geodesic

$\square$

**Remark:** One can also consider closed geodesics

$$\gamma : S^1 = \mathbb{R}/2\pi\mathbb{Z} \to M$$

without end points.




The $2^{nd}$ variation formula has important geometric and topological implications.

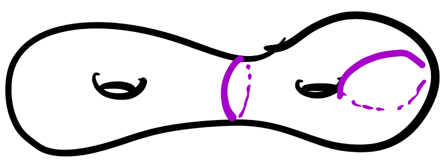
**Note:** $$E''(0) = \int_a^b \underbrace{\left\| \nabla_{\frac{\partial}{\partial t}} V\right\|^2}_{\geq 0} - \underbrace{\left\langle R\left(\frac{\partial\gamma}{\partial t}, V\right)\frac{\partial\gamma}{\partial t}, V\right\rangle}_{\text{sectional curvature term}} dt$$

**Cor**: Suppose $(M^n, g)$ has negative sectional curvature, i.e. $K < 0$.

THEN, any geodesic $\gamma : [a,b] \to M$ is (strictly) locally energy / length minimizing (with end points fixed).

i.e. any critical pt of $E$ or $L$ must be local minimum.

**E.g.)** On a hyperbolic surface $(\Sigma^2_{g \geq 2}, g_{hyp})$ of $K \equiv -1$.



For positively curved space, we have the following:

**Synge Theorem**: Suppose $(M^n, g)$ is a compact, oriented Riem. manifold s.t. (i) $n$ is even

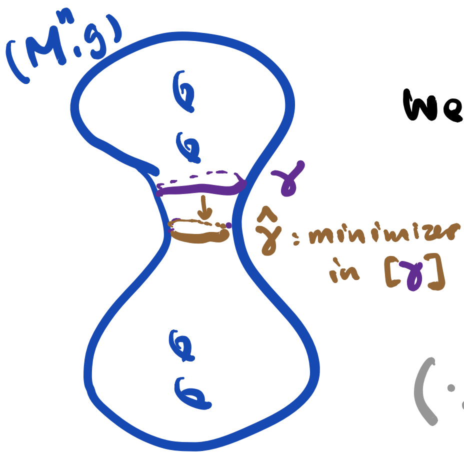
(ii) $\boxed{K > 0}$ everywhere

THEN, $\pi_1(M) = 0$, i.e. $M$ is simply-connected.

**Proof**: Suppose NOT. i.e $\pi_1(M) \neq 0$.

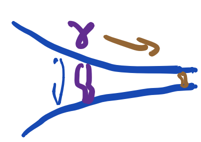
So, $\exists$ a (smooth) closed loop $\gamma : S^1 \to M$ which is NOT contractible to a pt. inside $M$, i.e $0 \neq [\gamma] \in \pi_1(M)$

free homotopy class




We want to do a minimization (w.r.t $E$) with the free homotopy class $[\gamma] \neq 0$

$$\min\{E(\tilde{\gamma}) : \tilde{\gamma} \in [\gamma]\} = E(\hat{\gamma})$$




($\because M$ cpt $\Rightarrow$ existence of minimizer $\hat{\gamma}$)

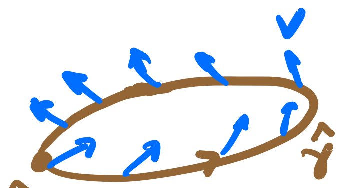
Note that $\hat{\gamma}$ is a non-trivial loop since $[\hat{\gamma}] = [\gamma] \neq 0$.

<u>AND</u>: $E'(0) = 0$ & $E''(0) \geq 0$ at $\hat{\gamma}$

for <u>any</u> variation field $V$ along $\hat{\gamma}$

<u>Write</u>: $\hat{\gamma} : [0, 2\pi]/_{0\sim 2\pi} \to M$ geodesic

<u>GOAL</u>: Find a $V$ st $E''(0) < 0$




$\hat{\gamma}(0) = \hat{\gamma}(2\pi) = p$

Let $P : T_pM \xrightarrow{\text{linear}} T_pM$ be the parallel transport map along $\hat{\gamma}$ from $\underbrace{\hat{\gamma}(0)}_{=p}$ to $\underbrace{\hat{\gamma}(2\pi)}_{=p}$

$\hat{\gamma}$ geodesic $\Leftrightarrow$ $\hat{\gamma}'$ is parallel along $\hat{\gamma}$

$\Rightarrow$ $P(\hat{\gamma}'(0)) = \hat{\gamma}'(0)$

Since $P$ preserve the inner product, we have

$$P : (\hat{\gamma}'(0))^{\perp} \xrightarrow[\cong]{\text{linear}} (\hat{\gamma}'(0))^{\perp} \quad \text{ie } P \in SO(n-1)$$

dim $M$ even $\Rightarrow$ $(\hat{\gamma}'(0))^{\perp}$ is odd dimensional

$\Rightarrow$ $\overset{0}{\exists}{}^{*} w \in (\hat{\gamma}'(0))^{\perp}$ st $P(w) = w$

Let $V(t)$ be the unique <u>parallel</u> v.f. along $\hat{\gamma}$ $\to$ $\nabla_{\frac{\partial}{\partial t}} V \equiv 0$

st $V(0) = V(2\pi) = w$

For this $V$,

$$0 > E''(0) = \int_0^{2\pi} \| \cancelto{0}{\nabla_{\frac{\partial}{\partial t}} V} \|^2 - \underbrace{\langle R(\tfrac{\partial \hat{\gamma}}{\partial t}, V)\tfrac{\partial \hat{\gamma}}{\partial t}, V \rangle}_{-\underbrace{K(\mathrm{span}\{V, \frac{\partial \hat{\gamma}}{\partial t}\})}_{<0}} \, dt$$

$\because$ $V$ parallel

Contradiction!

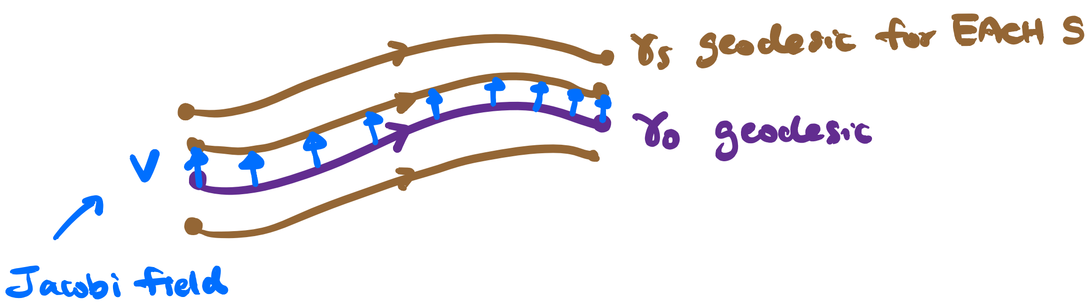
∎

Hopf Conjecture: Does $S^2 \times S^2$ admit a metric with $K > 0$ ?

# Jacobi Fields

Jacobi fields
- variation field for family of geodesics
- "linearized" geodesic equations.
- kernel of "index form", ie Hessian of $E$

## Motivations




E.g.) In $\mathbb{R}^2$


**Def$^{\underline{n}}$**: Given a geodesic $\gamma(t) : [a,b] \to M$, a vector field $V(t)$, $t \in [a,b]$, along $\gamma$ is said to be a Jacobi field if

$$\text{(J)} \qquad \boxed{\nabla_{\gamma'} \nabla_{\gamma'} V + R(\gamma', V)\gamma' = 0} \qquad \text{Jacobi field eq}^{\underline{n}}$$

**Def$^{\underline{n}}$**: The index form of a geodesic $\gamma : [a,b] \to M$ is

$$I(V, W) := \int_a^b \langle \nabla_{\gamma'} V, \nabla_{\gamma'} W \rangle - \langle R(\gamma', V)\gamma', W \rangle \, dt$$

**Note:** $I(V,V) = E''(0)$ along the variation field $V$

Symmetry of $R \Rightarrow I(V,W)$ is a symmetric bilinear form.

**Prop:** Let $V$ be a Jacobi field along a geodesic $\gamma : [a,b] \to M$.

THEN. $V \in$ "$\ker(I)$", ie

$$I(V,W) = 0 \quad \forall\, W \text{ v.f. along } \gamma \text{ s.t. } W(a) = 0 = W(b)$$

In fact, the converse also hold.

**Proof:** Recall the index form

$$I(V,W) := \int_a^b \langle \nabla_{\gamma'} V, \nabla_{\gamma'} W \rangle - \langle R(\gamma', V)\gamma', W \rangle \, dt$$

Integrate by part, using $W$ vanishes at the end pts.

$$I(V,W) := \int_a^b - \langle \nabla_{\gamma'} \nabla_{\gamma'} V, W \rangle - \langle R(\gamma', V)\gamma', W \rangle \, dt$$

$$= - \int_a^b \langle \underbrace{\nabla_{\gamma'} \nabla_{\gamma'} V + R(\gamma', V)\gamma'}_{=0 \;\Leftrightarrow\; (J)}, W \rangle \, dt$$

$\square$

Suppose $\gamma_s : [a,b] \to M$ is a geodesic for EACH $s \in (-\varepsilon, \varepsilon)$

i.e. $\forall\, s \in (-\varepsilon, \varepsilon)$, $\boxed{\nabla_{\gamma_s'} \gamma_s' \equiv 0}$ ← non-linear $2^{nd}$ order ODE system.

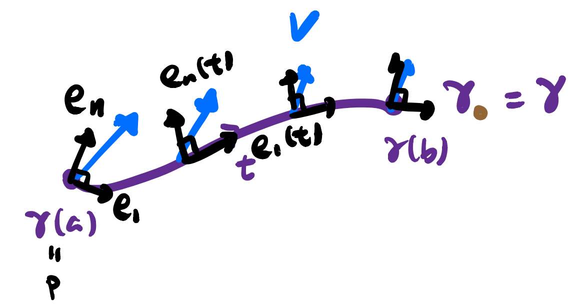
**IDEA:** If we differentiate the geodesic eq$^{n}$ w.r.t. $s$ at $s = 0$.

then we obtain the Jacobi field eq$^{n}$ (J) for $V := \left.\frac{\partial \gamma}{\partial s}\right|_{s=0}$.

(Ex: Prove this!)

**Lemma:** (J) is a $2^{nd}$ order LINEAR ODE system.

**Proof:**




Fix O.N.B. $\{e_1, \dots, e_n\}$ of $T_pM$

Parallel transport along $\gamma$

$\Longrightarrow$ obtain O.N.B. $\{e_1(t), \dots, e_n(t)\}$ for $T_{\gamma(t)}M$

Write $V(t) = \sum_{i=1}^{n} a_i(t)\, e_i(t)$. ← parallel

for some fcn $a_i(t)$.

$$(J) \qquad \nabla_{\gamma'}\nabla_{\gamma'}V + R(\gamma', V)\gamma' = 0$$

$$\Longleftrightarrow \quad \sum_{i=1}^{n} a_i''(t)\, e_i(t) + \sum_{i=1}^{n} a_i(t) \underbrace{R(\gamma', e_i(t))\gamma'}_{\sum_{j=1}^{n} \langle R(\gamma', e_i(t))\gamma',\, e_j(t)\rangle e_j(t)} = 0$$

$$\Longleftrightarrow \quad \underbrace{a_i''(t) + \sum_{j=1}^{n} a_i(t)\, R(\gamma', e_j, \gamma', e_i) = 0}_{2^{nd} \text{ order linear system in } a_i(t)} \qquad \forall\, i$$

**Cor:** (J) is uniquely solvable on $[a,b]$ for any given initial data $V(a)$ and $V'(a) := (\nabla_{\gamma'}V)(a)$.

depends linearly on initial data

**Note that:** Any vector field $V$ along $\gamma$ decompose:

$$V = \underset{\substack{\uparrow \\ \text{tangent} \\ \text{to } \gamma}}{V^{T}} + \underset{\substack{\uparrow \\ \text{normal} \\ \text{to } \gamma}}{V^{\perp}}$$

**Prop:** Any tangential (i.e. $V^\perp \equiv 0$) Jacobi field $V$ along $\gamma$ has the form

$$V(t) = \underbrace{\big(A + B(t-a)\big)}_{\text{linear in } t}\gamma'(t) \quad \text{for some constants } A, B \in \mathbb{R}$$

**Pf:** Solving uniquely (J) with initial data

$$V(a) = A\,\gamma'(a) \quad \text{and} \quad V'(a) = B\,\gamma'(a).$$

$\blacksquare$

**Remark:** This implies "tangential" Jacobi fields are NOT useful, but the "normal" Jacobi fields contains a lot of information about the geometry of $(M^n, g)$.

**Prop:** Suppose $\gamma_s : [a,b] \to M$ is a 1-parameter family s.t. $\gamma_s$ is a geodesic for EACH $s \in (-\varepsilon, \varepsilon)$.

THEN, the variation field $V := \left.\dfrac{\partial \gamma}{\partial s}\right|_{s=0}$ satisfies (J).

**Remark:** The converse is also true, i.e. Jacobi fields along geodesics are all "integrable". (Pf. HW)

**Proof:** Each $\gamma_s$ is a geodesic

$$\Rightarrow \quad \nabla_{\gamma_s'} \gamma_s' \equiv 0 \qquad \forall\, s \in (-\varepsilon, \varepsilon)$$

$$\Leftrightarrow \quad \boxed{\nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \equiv 0} \quad (*) \qquad \forall\, s \in (-\varepsilon, \varepsilon).$$

Consider

$$\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s} \overset{\text{torsion-free}}{=} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t}$$

$$= \nabla_{\frac{\partial}{\partial s}} \overbrace{\nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t}}^{(*)=0} + R\left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}\right) \frac{\partial \gamma}{\partial t}$$

Evaluate at $s=0$,

$$\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} V + R\left(\frac{\partial \gamma}{\partial t}, V\right) \frac{\partial \gamma}{\partial t} = 0 \qquad (J)$$

∎